Illustration - 26 *A mixed doubles tennis game is to be arranged from 5 married couples. In how many ways the game be arranged if no husband and wife pair is included in the same game?*

(A) 60 **(B)** 30 **(C)** 120 **(D)** 240

SOLUTION : (A)

To arrange the game we have to do the following operations.

(i) Select two men from 5 men in ${}^5C_2$ ways.

(ii) Select two women from 3 women excluding the wives of the men already selected. This can be done in ${}^3C_2$ ways.

(iii) Arrange the 4 selected persons in two teams. If the selected men are M_1 and M_2 and the selected women are W_1 and W_2 , this can be done in 2 ways :

M_1W_1 play against M_2W_2

M_2W_1 play against M_1W_2

Hence, the number of ways to arrange the game $= {}^5C_2 \; {}^3C_2 \; (2) = 10 \times 3 \times 2 = 60$

TYPICAL PROBLEM CATEGORIES — SECTION - 5

Most of the typical problems in permutation and combination can be categorised in various types. To solve these typical problems, it is advisable to apply standard ways or methods available.

Let us call these categories of problems as Typical Problem Categories (TPC).

5.1 TPC - 1 : Always including particular objects in the selection

The number of ways to select r objects from n different objects where p particular objects should always be included in the selection $= {}^{n-p}C_{r-p}$.

Logic :

We can select p particular objects in 1 way. Now from remaining $(n - p)$ objects we select remaining $(r - p)$ objects in ${}^{n-p}C_{r-p}$ ways.

Using fundamental principle of counting, number of ways to select r objects where p particular objects are always included $= 1 \times {}^{n-p}C_{r-p} = {}^{n-p}C_{r-p}$.

***Illustrating the Concepts* :**

In how many ways a team of 11 players be selected from a list of 16 players where two particular players should always be included in the team.

Using formula given in TPC-1, number of ways to make a team of 11 players from 16 players always including 2 particular players $= {}^{16-2}C_{11-2} = {}^{14}C_9$

5.2 TPC-2 : Always excluding p particular objects in the selection

The number of ways to select r objects from n different objects where p particular objects should never be included in the selection = ${}^{n-p}C_r$.

Logic :

As p particular objects are never to be selected, selection should be made from remaining $(n - p)$ objects. Therefore r objects can be selected from $(n - p)$ different objects in ${}^{n-p}C_r$ ways.

Illustrating the Concepts :

In how many ways a team of 11 players can be selected from a list of 16 players such that 2 particular never be included players should be included in the selection.

Using the formula given in TPC-2, the number of ways to select a team of 11 players from a list of 16 players, always excluding 2 particular players = ${}^{16-2}C_{11} = {}^{14}C_{11}$

5.3 TPC-3 : Always including p particular objects in the arrangement

The number of ways to select and arrange (permutate) r objects from n different objects such that arrangement should always include p particular objects = ${}^{n-p}C_{r-p}\ \underline{|r}$.

Logic :

First select p particular objects which should always be included in 1 way. **. . . (i)**

Then select remaining $(r - p)$ objects from remaining $(n - p)$ objects in ${}^{n-p}C_{r-p}$ ways. **. . . (ii)**

Finally arrange r selected objects in $\underline{|r}$ ways. **. . . (iii)**

Using fundamental principle of counting, operations (i), (ii) and (iii) can be performed together in ways

$$= 1 \times {}^{n-p}C_{r-p} \times \underline{|r} \text{ ways.}$$

5.4 TPC-4 : Always excluding p particular objects in the arrangement

The number of ways to select and arrange r objects from n different objects such that p particular objects are always excluded in the selection = ${}^{n-p}C_r\ \underline{|r}$.

Logic :

First exclude p particular objects from n different objects.

Then select r objects from $(n - p)$ different objects in ${}^{n-p}C_r$ ways. **. . . (i)**

Then permutate r selected objects in $\underline{|r}$ ways. **. . . (ii)**

Using fundamental principle of counting, operations (i) and (ii) can be performed together in ways

$$= {}^{n-p}C_r\underline{|r} \text{ ways.}$$

***Illustrating the Concepts* :**

How many three-letter words can be made using the letters of the words 'SOCIETY', so that

(i) *'S' is included in each word?* **(ii)** *'S' is not included in any word?*

(i) To include S in every word, we have two cases.

Step - I :

Select the remaining two letters from remaining 6 letters i.e. O, C, I, E, T, Y in ${}^{7-1}C_{3-1} = {}^6C_2$ ways.

Step - II :

Include S in each group and then arrange each group of three in 3! ways.

$\Rightarrow$ Number of words = ${}^6C_2\ 3! = 90$

(ii) If S is not to be included, then we have to make all the three words from the remaining 6.

$\Rightarrow$ Number of words = ${}^6C_3\ 3! = 120$

5.5 TPC-5 : *p* particular objects always together in the arrangement.

The number of ways to arrange n different objects such that p particular objects remain together in the arrangement

$$= \lfloor(n-p+1)\ \lfloor p$$

Logic :

Make a group of p particular objects that should remain together. Arrange this group of p particular objects and remaining $(n-p)$ objects in $\lfloor n-p+1$ ways. . . . (i)

Finally arrange p particular objects amongst themselves in $\lfloor p$ ways. . . . (ii)

Using fundamental principle of counting, operations (i) and (ii) can be performed together in ways

$$= \lfloor(n-p+1)\ \lfloor p.$$

***Illustrating the Concepts* :**

How many words can be formed using the letters of the word 'TRIANGLE' so that

(a) *'A' and 'N' are always together ?* **(b)** *'T', 'R', 'I' are always together ?*

(a) Assume (AN) as a single letter. Now there are seven letters in all : $(AN), T, R, I, G, L, E$

Seven letters can be arranged in 7 ! ways

All these 7! words will contain A and N together. A and N can now be arranged amongst themselves in 2! ways (AN and NA).

Hence total number of words = 7! 2! = 10080

(b) Assume (TRI) as a single letter.

- The letters : $(TRI), A, N, G, L, E$ can be arranged in 6! ways.
- TRI can be arranged amongst themselves in 3! ways.

Total number of words = 6! 3! = 4320

Illustration - 27 *How many five-letter words containing 3 vowels and 2 consonants can be formed using the letters of the word 'EQUATION' so that the two consonants occur together in every word?*

(A) 240 **(B)** 1440 **(C)** 720 **(D)** 480

SOLUTION : (B)

There are 5 vowels and 3 consonants in the word 'EQUATION'. To form the words, we have three cases.

Step I : Select vowels (3 from 5) in ${}^5C_3$ ways.

Step II : Select consonants (2 from 3) in ${}^3C_2$ ways.

Step III : Arrange the selected letters (3 vowels and 2 consonants (always together)) in $4\,! \times 2\,!$ ways.

Hence the no. of words = ${}^5C_3\ {}^3C_2\ 4\,!\ 2\,!$

$= 10 \times 3 \times 24 \times 2 = 1440$

5.6 TPC-6 : p particular objects always separated in the arrangement.

The number of ways to arrange n different objects such that p particular objects are always separated

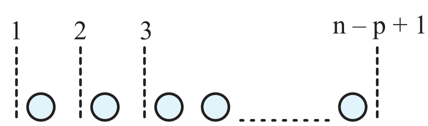
$$= {}^{n-p+1}C_p\ \underline{|n-p}\ \underline{|p}$$

Logic :

We have to place p particular objects between $(n-p)$ remaining objects so that all p particular objects must be separated from each other.

From figure, we can see there are $(n-p+1)$ places between $(n-p)$ objects where we can place p particular objects such that p objects are separated from each other.

Select p places from $(n-p+1)$ places for p particular objects in ${}^{n-p+1}C_p$ ways.



(n – p) remaining objects

Now place and arrange p objects in p selected places in $\underline{|p}$ ways. If all p particular objects are not different, then we use formula given in 3.4 to arrange p objects.

Finally, arrange $(n-p)$ remaining objects in $\underline{|n-p}$ ways. If $(n-p)$ objects are not different, then use formula given in section 3.4 to arrange them.

Using fundamental principle of counting, all operations can be done together in ${}^{n-p+1}C_p\ \underline{|p}\ \underline{|n-p}$ ways.

Illustrating the Concepts **:**

There are 9 candidates for an examination out of which 3 are appearing in Mathematics and remaining 6 are appearing in different subjects. In how many ways can they be seated in a row so that no two Mathematics candidates are together ?

Divide the work in two cases.

Step - I :

First, arrange the remaining candidates in 6 ! ways.

Step - II :

Place the three Mathematics candidates in the row of six other candidates so that no two of them are together.

X : Places available for Mathematics candidates.

O : Others.

X	O	X	O	X	O	X	O	X	O	X	O	X

In any arrangement of 6 other candidates (O), there are seven places available for Mathematics candidates so that they are not together. Now, 3 Mathematics candidates can be placed in these 7 places in ${}^7P_3$ ways.

Hence total number of arrangements

$$= 6!\,{}^7P_3 = 720 \times \frac{7!}{4!} = 151200$$

Illustration - 28 *In how many ways can 7 plus (+) signs and 5 minus (–) signs be arranged in a row so that no two minus (–) signs are together?*

(A) ${}^8C_5$ **(B)** ${}^8C_5 \times 7! \times 5!$ **(C)** ${}^8C_5 \times 5!$ **(D)** ${}^8C_5 \times 7!$

SOLUTION : (A)

Step - I :

The plus signs can be arranged in one way (because all are identical).

	+		+		+		+		+		+		+	

A blank box shows available spaces for the minus signs.

Step - II :

The 5 minus (–) signs are now to be placed in the 8 available spaces so that no two of them are together.

(i) Select 5 places for minus signs in ${}^8C_5$ ways.

(ii) Arrange the minus signs in the selected places in 1 way (all signs being identical).

Hence number of possible arrangements $= 1 \times {}^8C_5 \times 1 = 56$

5.7 TPC - 7 : Problems based on atleast or at most constraint

There are problems in which constraints are to select minimum (at least) or maximum (at most) objects in the selection.

In these problems, we should always make cases to select objects. If we don't make cases, we will get wrong answer. Following illustrations will show you how to make cases to solve problems of this type.

Illustrating the Concepts :

A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour ?

The selection of balls from 5 red and 6 white balls will consist of any of the following possibilities.

Red Balls (out of 5)	2	3	4
White Balls (out of 6)	4	3	2

- If the selection contains 2 red & 4 white balls, then it can be done in ${}^5C_2\ {}^6C_4$ ways.
- If the selection contains 3 red & 3 white balls then it can be done in ${}^5C_3\ {}^6C_3$ ways.
- If the selection contains 4 red & 2 white balls then it can be done in ${}^5C_4\ {}^6C_2$ ways.

Any one of the above three cases can occur. Hence the total number of ways to select the balls

$$= {}^5C_2\ {}^6C_4 + {}^5C_3\ {}^6C_3 + {}^5C_4\ {}^6C_2 = 10\,(15) + 10\,(20) + 5\,(15) = 425$$

Illustration - 29 *In how many ways a team of 5 members can be selected from 4 ladies and 8 gentlemen such that selection includes at least 2 ladies ?*

(A) 336 **(B)** 448 **(C)** 449 **(D)** 456

SOLUTION : (D)

As the selection includes 'at least' constraint, we make cases to find total number of teams.

Ladies in the team (4)	Gentlemen in the team (8)	Number of ways to select team of 5
2	3	${}^4C_2 \times {}^8C_3$
3	2	${}^4C_3 \times {}^8C_2$
4	1	${}^4C_4 \times {}^8C_1$

Combining all cases shown in the table = total number of ways to select a team of 5 members

$$= {}^4C_2 \times {}^8C_3 + {}^4C_3 \times {}^8C_2 + {}^4C_4 \times {}^8C_1 = 456$$

5.8 TPC-8 : Permutations of *n* objects taken *r* at a time when all *n* objects are not different

In this section, we will discuss how to arrange (permutate) n objects taken r at a time where all n objects are not different. For example, arrangements of letters $AABBBC$ taken 3 at a time.

To find such arrangements, it is not possible to derive a formula that can be applied in all such cases.

So,we will discuss a method (or procedure) that should be applied to find arrangements. The method involves making cases based on alike items that we choose in the arrangement. You should read the following illustrations to learn how to apply this "method of cases" to find arrangements of n objects taken r at a time when all objects are not different.

***Illustrating the Concepts* :**

In how many ways we can arrange letters A, A, B, B, B, C taken 3 at a time.

The given letters include AA, BBB, C *i.e.* $2A$ letters, $3B$ letters and 1 C letter.

To find arrangements of B letters, we will make following cases based on alike letters we choose in the arrangement.

Case - 1 : All 3 letters are alike

- 3 alike letters can be selected from given letters in only 1 way *i.e.* BBB.

Further 3 selected letters can be arranged amongst themselves in $\frac{\lfloor 3}{\lfloor 3} = 1$ way.

$\Rightarrow$ Total number of arrangements with all letters alike = 1 **. . . (i)**

Case - 2 : 2 alike and 1 different

- 2 alike letters can be selected from 2 sets of alike letters (AA, BB) in ${}^2C_1$ ways.
- 1 different letter (different from alike letters) can be selected from remaining letters in ${}^2C_1$ ways. (C, A or B either).

Further 2 alike and 1 different selected letters can be arranged amongst themselves in $\frac{\lfloor 3}{\lfloor 2}$ ways.

$\Rightarrow$ Total number of arrangements with "2 alike and 1 different letter" $= {}^2C_1 \times {}^2C_1 \times \frac{\lfloor 3}{\lfloor 2} = 2 \times 2 \times 3 = 12$ **. . .(ii)**

Case - 3 : All different letters

- All 3 letters different can be selected from 3 different letters (A, B, C) in 1 way.

Further 3 different letters can be arranged amongst themselves in $\lfloor 3$ ways.

$\Rightarrow$ Total number of arrangements with all 3 letters different $= 1 \times \lfloor 3 = \lfloor 3 = 6$ **. . . (iii)**

Combining (i), (ii) and (iii),

Total number of permutations of A, A, B, B, C taken 3 at a time $= 1 + 12 + 6 = 19$

Illustration - 30 *How many four-letter words can be formed using the letters of the word 'INEFFECTIVE'?*

(A) 840 **(B)** 1380 **(C)** 1422 **(D)** *None of these*

SOLUTION : (C)

'*INEFFECTIVE*' contains 11 letters : EEE, FF, II, C, T, N, V.

As all letters are not different, we cannot use ${}^nP_r$. The four-letter words will be from any one of the following categories.

1. 3 alike letters, 1 different letter.
2. 2 alike letters, 2 alike letters.
3. 2 alike letters, 2 different letters.
4. All different letters.

1. 3 alike, 1 different :

3 alike can be selected in one way i.e. EEE.

Different letters can be selected from F, I, T, N, V, C in ${}^6C_1$ ways.

$\Rightarrow$ Number of groups $= 1 \times {}^6C_1 = 6$ $\Rightarrow$ Number of words $= 6 \times \frac{4!}{3! \times 1!} = 24$

2. 2 alike, 2 alike :

Two sets of 2 alike can be selected from 3 sets (*EE, II, FF*) in ${}^3C_2$ ways.

$\Rightarrow$ Number of groups = ${}^3C_2$ $\Rightarrow$ Number of words = ${}^3C_2 \times \dfrac{4!}{2! \times 2!} = 18$

3. 2 alike, 2 different :

$\Rightarrow$ Number of groups = $({}^3C_1) \times ({}^6C_2) = 45$ $\Rightarrow$ Number of words = $45 \times \dfrac{4!}{2!} = 540$

4. All different :

$\Rightarrow$ Number of groups = ${}^7C_4$ (out of E, F, I, T, N, V, C) = 35 $\Rightarrow$ Number of words = $35 \times 4! = 840$

Hence, total four-letter words = 24 + 18 + 540 + 840 = 1422

5.9 TPC-9 : Selection of *r* objects from *n* objects when all *n* objects are not different.

In this problem type, we will discuss how to select r objects from n objects when all n objects are not different.

For example, selection of 3 letters from letters $AABBBC$.

To find number of ways to select, it is not possible to derive a formula that can be applied in all such cases.

Instead of formula, we will discuss a method (procedure) that should be applied to find selections.

The method involves making cases based on alike items in the selection. You should be through the following illustrations to learn how to apply this "method of cases" to find selections of r objects from n objects when all n objects are not different.

Illustrating the Concepts :

In how many ways 3 letters can be selected from letters A, A, B, B, B, C.

The given letters include AA, BBB, C *i.e.* $2A$ letters, $3B$ letters and $1C$ letter.

To find number of selections, we will make the following cases based on alike letters we choose in the selection.

Case - 1 : All 3 letters are alike

- 3 alike letters can be selected from given letters in only 1 way *i.e.* BBB.

$\Rightarrow$ The number of selections with all 3 letters alike = 1 . . . **(i)**

Case - 2 : 2 alike and 1 different letter

- 2 alike letters can be selected from 2 sets of alike letters (AA, BB) in ${}^2C_1$ ways.
- 1 different letter (different from alike letters) can be selected from remaining letters in ${}^2C_1$ ways. (either A or B).

Using fundamental principle of counting,

Total number of selections with 2 alike and 1 different letter = ${}^2C_1 \times {}^2C_1 = 4$ ways . . . **(ii)**

Case - 3 : All letters different

- All 3 letters different can be selected from 3 different letters (A, B, C) in 1 way.

$\Rightarrow$ Total number of ways to select 3 different letters is 1 way . . . **(iii)**

Combining (i), (ii) and (iii),

Total number of ways to select 3 letters from given letters = 1 + 4 + 1 = 6.

Illustration - 31 *In how many ways 4 letters can be selected from the letters of the word 'INEFFECTIVE' ?*

(A) 80 **(B)** 89 **(C)** 51 **(D)** *None of these*

SOLUTION : (B)

INEFFECTIVE contains 11 letters : *EEE, FF, II, C, T, N, V*

We will make following cases to select 4 letters.

Case - 1 : 3 alike and 1 different

- 3 alike letters can be selected from 1 set of 3 alike letters (*EEE*) in 1 way.

$\Rightarrow$ The number of ways to select 3 alike letters = 1

$\Rightarrow$ The number of ways to select 1 different letters $= 6 \Rightarrow$ Total ways $= 6 \times 1 = 6$ **. . . (i)**

Case - 2 : 2 alike and 2 alike

- '2 alike and 2 alike' means we have to select 2 groups of 2 alike letters (*EE, FF, II*) in ${}^3C_2$ ways.

$\Rightarrow$ The number of ways to select "2 alike and 2 alike" letters $= {}^3C_2 = 3$.

Case - 3 : 2 alike and 2 different

- 1 group of 2 alike letters can be selected from 3 sets of 2 alike letters (*EE, FF, II*) in ${}^3C_1$ ways.
- 2 different letters can be selected from 6 different letters (*C, T, N, V,* remaining 2 sets of two letters alike) in ${}^6C_2$ ways.

$\Rightarrow$ The number of ways to select "2 alike and 2 different letters" ${}^3C_1 \times {}^6C_2 = 3 \times 15 = 45$ **. . . (ii)**

Case - 4 : All different letters

- All different letters can be selected from 7 different letters (*I, E, F, N, C, T, V*) in ${}^7C_4$ ways.

$\Rightarrow$ The number of ways to select all different letters $= {}^7C_4 = 35$ **. . . (iii)**

Combining (i), (ii), (iii), we get

Total number of ways to select 4 letters from the letters of the word *INEFFECTIVE* $= 6 + 3 + 45 + 35 = 89$.

5.10 TPC-10 : Selection of one or more objects

(A) Selection of one or more objects from *n* different objects

The number of ways to select one or more objects from n different objects or we can say, selection of at least one object from n different objects $= 2^n - 1$.

Logic :

The number of ways to select 1 object from n different objects $= {}^nC_1$

The number of ways to select 2 objects from n different objects $= {}^nC_2$

- - - - - -   - - - - - -   - - - - - -

- - - - - -   - - - - - -   - - - - - -

- - - - - -   - - - - - -   - - - - - -

The number of ways to select n objects from n different objects $= {}^nC_n$

Combining all above cases, we get

The number of ways to select at least one (one or more) object from n different objects

$= {}^nC_1 + {}^nC_2 + {}^nC_3 + {}^nC_4 + \ldots\ldots + {}^nC_n$

$= 2^n - 1$ [Using sum of binomial coefficients in the expansion of $(1 + x)^n = 2^n$]

Alternate Logic :

Let us assume $a_1, a_2, a_3, \ldots\ldots, a_n$ be n different objects.

We have to make our selection from these n objects.

Objects	a_1	a_2	a_3	a_4	. . . . . .	a_n
Ways	2	2	2	2	. . . . . .	2

We can make our selection from a_1 object in 2 ways.

This is because either we will choose a_1 or we would not choose a_1. Similarly, selection of $a_2, a_3, \ldots\ldots, a_n$ can be done in 2 ways each.

Using fundamental principle of counting,

The total number of ways to make selection of at least one object from $a_1, a_2, \ldots\ldots, a_n$

$= 2 \times 2 \times 2 \times 2 \ldots\ldots\ldots n$ times

$= 2^n$

But the above selection includes a case where we have not selected any object. On subtracting this case from 2^n we get, the number of ways to select atleast one (one or more) object from n different objects $= 2^n - 1$

Note :

(a) The number of ways to select 0 or more objects from n different objects $= 2^n$

(b) The number of ways to select at least 2 objects from n different objects $= 2^n - 1 - {}^nC_1$

(c) The number of ways to select at least r objects from n different objects

$= 2^n - 1 - {}^nC_1 - {}^nC_2 - {}^nC_3 - \ldots\ldots - {}^nC_{r-1}$

(B) Selection of one or more objects from n identical objects

The number of ways to select one or more objects (or at least one object) from n identical objects $= n$

Logic :

To select r objects from n identical objects, we can not use ${}^nC_r$ formula here as all objects are not different. In fact, all objects all identical. It means we can not choose objects. It does not matter which r objects we take as all objects are identical.

The number of ways to select 1 object from n identical objects = 1

The number of ways to select 2 objects from n identical objects = 1

- - - - - - - - - - - - - - - - - -

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- - - - - - - - - - - - - - - - - -

The number of ways to select n objects from n identical objects = 1

Combining all above cases, we get

Total number of ways to select 1 or more objects from n identical objects

$= 1 + 1 + \ldots\ldots n$ times $= n$

Note :

(a) The number of ways to select 0 or more objects from n identical objects $= n + 1$

(b) The number of ways to select at least 2 objects from n identical objects $= n - 1$

(c) The number of ways to select r objects from n identical objects is 1

(d) The total number of selections of some or all out of $(p + q + r)$ objects where p are alike of one kind, q are alike of second kind and rest r are alike of third kind is

$(p + 1)(q + 1)(r + 1) - 1$ [Using fundamental principle of counting]

(C) Selection of one or more objects from objects which are not all different from each other.

The number of ways to select one or more objects from $(p + q + r \ldots\ldots + n)$ objects where p objects are alike of one kind, q are alike of second kind, r are alike of third kind, ... and remaining n are different from each other

$= [(p + 1)(q + 1)(r + 1) \ldots\ldots. 2^n] - 1.$

Logic :

The number of ways to select 0 or more objects from p alike objects of one kind $= p + 1$

The number of ways to select 0 or more objects from q alike objects of second kind $= q + 1$

The number of ways to select 0 or more objects from r alike objects of third kind $= r + 1$

- - - - - -   - - - - - -   - - - - - -
- - - - - -   - - - - - -   - - - - - -
- - - - - -   - - - - - -   - - - - - -

The number of ways to select 0 or more objects from n different objects $= 2^n$

Combining all cases and using fundamental principle of counting, we get :

Total number of ways to select 0 or more objects $= [(p + 1)(q + 1)(r + 1) \ldots\ldots 2^n]$ ... (i)

But above selection includes a case where we have not selected any object. So we need to subtract 1 from the above result if we want to select at least one object.

Therefore, the total number of ways to select one or more objects (at least one) from p alike of one kind, q alike of another kind, r alike of third kind, ..., and n different objects $= [(p + 1)(q + 1)(r + 1) \ldots\ldots 2^n] - 1$

Note :

(a) The number of ways to select 0 or more objects from p alike of one kind, q alike of second kind, r alike of third kind and n different objects $= (p + 1)(q + 1)(r + 1)\, 2^n$

(b) The number of ways to select objects from p alike of one kind, q alike of second kind and r alike of third kind and n different objects such that selection includes atleast one object of each of the three kinds of alike objects and atleast one of the n different objects.

$= pqr\,(2^n - 1)$

Illustrating the Concepts :

Find the number of ways in which one or more letters can be selected from the letters :

$A, A, A, A, B, B, B, C, D, E.$

The given letters can be divided into five following categories : $(AAAA)$, (BBB), C, D, E.

To select at least one letter, we have to take five decisions - one for every category.

Selections from $(AAAA)$ can be made in 5 ways :

Include no A, include one A, include AA, include AAA, include $AAAA$.

Similarly, selections from (BBB) can be made in 4 ways, and selections from C, D, E can be made in $2 \times 2 \times 2$ ways.

$\Rightarrow$ Total number of selections

$= 5 \times 4 \times (2 \times 2 \times 2) - 1 = 159$

(excluding the case when no letter is selected)

Illustration - 32 *A man has 5 friends. In how many ways can he invite one or more of them to a party?*

(A) 32 **(B)** 31 **(C)** 30 **(D)** 16

SOLUTION : (B)

If he invites one person to the party,

Number of ways $= {}^5C_1$

If he invites two persons to the party,

Number of ways $= {}^5C_2$

Proceeding on the similar pattern,

Total number of ways to invite

$$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$$

$$= 5 + 10 + 10 + 5 + 1 = 31$$

Alternate Approach :

To invite one or more friends to the party, he has to take 5 decisions – one for every friend.

Each decision can be taken in two ways - invited or not invited.

Hence the number of ways to invite one or more

= (number of ways to make 5 decisions – 1)

$= 2 \times 2 \times 2 \times 2 \times 2 - 1 = 2^5 - 1 = 31$

Note that we have to subtract 1 to exclude the case when all are not invited.

Illustration - 33 *The question paper in the examination contains three sections - A, B, C. There are 6, 4, 3 questions in sections A, B, C respectively. A student has the freedom to answer any number of questions attempting at least one from each section. In how many ways can the paper be attempted by a student?*

(A) 8192 **(B)** 6615 **(C)** 7168 **(D)** *None of these*

SOLUTION : (B)

There are three possible cases :

Case - I :

Section A contains 6 questions.The student can select at least one from these in $2^6 - 1$ ways.

Case - II :

Section B contains 4 questions. The student can select at least one from these in $2^4 - 1$ ways.

Case - III :

Section C can similarly be attempted in $2^3 - 1$ ways.

Hence, total number of ways to attempt the paper

$$= (2^6 - 1)(2^4 - 1)(2^3 - 1)$$

$$= 63 \times 15 \times 7 = 6615$$

Illustration - 34 *Find the number of factors (excluding 1 & the expression itself) of the product of $a^7\ b^4\ c^3\ d\ e\ f$ where a, b, c, d, e, f are all prime numbers.*

(A) 1280 **(B)** 1279 **(C)** 1278 **(D)** 1260

SOLUTION : (C)

A factor of expression $a^7\ b^4\ c^3\ d\ e\ f$ is simply the result of selecting one or more letters from 7 a's, 4 b's, 3 c's, d, e, f.

The collection of letters can be observed as a collection of 17 objects out of which 7 are alike of one kind (a's), 4 are of second kind (b's), 3 are of third kind (c's) and 3 are different (d, e, f).

The number of selections

$$= (1+7)(1+4)(1+3)\,2^3 = 8 \times 5 \times 4 \times 8 = 1280.$$

But we have to exclude two cases :

(i) When no letter is selected,

(ii) When all letters are selected.

Hence the number of factors = 1280 – 2 = 1278

5.11 TPC-11 : Derangement Theorem

If n distinct objects are to be arranged in a row such that no object occupies its original place, then to find number of ways to arrange them, we use derangement theorem *i.e.*

$$\text{Number of ways to derange} = \lfloor n \left[1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \ldots\ldots (-1)^n \frac{1}{\lfloor n}\right]$$

Let S_1, S_2, S_3 are three slots where objects A, B, C should be placed.

Number of ways to place A, B, C in S_1, S_2, S_3 such that A goes to S_1, B goes to S_2 and C goes to S_3 i.e. all objects are placed in their correct places = 1.

Number of ways to place only one object in a wrong slot is not possible because if A is placed in say S_2, then B, whose correct slot is S_2, would take either S_1 or S_3. It means B is also placed in the wrong slot. So it is not possible to place one object in wrong slot.

To place objects A, B, C in S_1, S_2, S_3 such that all objects are placed in wrong slots, we use derangement theorem *i.e.*

$$\text{Number of ways to place } A, B, C \text{ all in wrong slots} = \lfloor 3 \left[1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3}\right] = 2 \text{ ways.}$$

Illustration - 35 *There are 5 boxes of 5 different colors. Also there are 5 balls of colors same as those of the boxes. In how many ways we can place 5 balls in 5 boxes such that*

(i) *all balls are placed in the boxes of colors not same as those of the ball.*

(A) 44 **(B)** 45 **(C)** 48 **(D)** 60

(ii) *at least 2 balls are placed in boxes of the same color.*

(A) 32 **(B)** 31 **(C)** 76 **(D)** 75

SOLUTION : (i).(A) (ii).(B)

(i) All the balls should be placed in the wrong boxes

i.e. boxes not of the color same as balls.

Using derangement theorem, number of ways in which this can be done.

$$= \lfloor 5 \left[1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \frac{1}{\lfloor 5}\right]$$

$$= 120\left[1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right]$$

$= 60 - 20 + 5 - 1 = 44$

(ii) Atleast 2 balls are placed in the correct boxes i.e. boxes of the color same as ball

= Total number of ways to place balls in boxes – No. of ways to place balls such that all balls are placed in wrong boxes – No. of ways to place balls in boxes such that 1 ball is placed in the correct box (i.e. boxes of the same color as balls).

= $\lfloor 5 - 44 -$ No. of ways to select a ball that will be in correct box × No. of ways in which remaining 4 balls can be placed in 4 boxes such that all balls go in wrong boxes (boxes of color different from balls).

$$= \lfloor 5 - 44 - {}^5C_1 \times \lfloor 4 \left[1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4}\right]$$

$= 120 - 44 - 5 \times 9$ [Using answer of (i) part and derangement theorem]

$= 120 - 44 - 45 = 31$

5.12 TPC-12 : Sum of the numbers

In this TPC, we will learn how to find sum of all the numbers that can be formed using the given digits.

The following illustrations will help you learn how to find the sum of the numbers.

Illustrating the Concepts :

Find the sum of all five-digit numbers that can be formed using digits 1, 2, 3, 4, 5 *if repetition is not allowed?*

There are 5! = 120 five digit numbers and there are 5 digits. Hence by symmetry or otherwise, we can see that each digit will appear in any place *i.e.*

(unit's or ten's or . . . .) $\frac{5!}{5}$ times.

$\Rightarrow$ X = sum of digits in any place

$$\Rightarrow \quad X = \frac{5!}{5}\times 5 + \frac{5!}{5}\times 4 + \frac{5!}{5}\times 3 + \frac{5!}{5}\times 2 + \frac{5!}{5}\times 1$$

$$\Rightarrow \quad X = \frac{5!}{5}\times(5+4+3+2+1) = \frac{5!}{5}(15)$$

$\Rightarrow$ Hence, the sum of all numbers

$= X + 10X + 100X + 1000X + 10000X$

$= X(1 + 10 + 100 + 1000 + 10000)$

$$= \frac{5!}{5}(15)(1+10+100+1000+10000)$$

$= 24\,(15)\,(11111) = 3999960$

5.13 TPC-13 : Rank of a word in the dictionary

In this problem type, dictionary of words is formed by using the letters of the given word. The dictionary format means words are arranged in the alphabetical order. You will be supposed to find the rank (position) of the given word or some other word in the dictionary.

Following illustrations will help you learn how to find the rank in the dictionary.

Illustrating the Concepts :

If all the letters of the word 'RANDOM' are written in all possible orders and these words are written out as in a dictionary, then find the rank of the word 'RANDOM' in the dictionary.

In a dictionary, the words at each stage are arranged in alphabetical order. In the given problem, we must therefore consider the words beginning with A, D, M, N, O, R in order. A will occur in the first place as often as there are ways of arranging the remaining 5 letters all at a time i.e. A will occur 5! times. D, M, N, O will occur in the first place the same number of times.

Number of words starting with $A = 5! = 120$

Number of words starting with $D = 5! = 120$

Number of words starting with $M = 5! = 120$

Number of words starting with $N = 5! = 120$

Number of words starting with $O = 5! = 120$

After this, words beginning with RA must follow.

Number of words beginning with RAD or $RAM = 3!$

Now the words beginning with RAN must follow.

First one is *RANDMO* and the next one is *RANDOM*

$\therefore$ Rank of $RANDOM = 5\,(5!) + 2\,(3!) + 2 = 614$

Illustration - 36 *Find the rank of the word 'TTEERL' in the dictionary of words formed by using the letters of the word 'LETTER'.*

(A) 168 **(B)** 170 **(C)** 169 **(D)** 171

SOLUTION : (B)

In the dictionary of words formed, we need to count words before the word '*TTEERL*' in the dictionary. To count such words, we need to first count words starting with E, L, R, TE, TL, TR and then add 2 to the count for words '*TTEELR*' and '*TTEERL*'.

Number of words starting with E = Arrangement of letters $E, T, T, R, L = \dfrac{\underline{|5}}{\underline{|2}}$

Number of words starting with L = Arrangement of letters $E, T, T, E, R = \dfrac{\underline{|5}}{\underline{|2}\,\underline{|2}}$

Number of words starting with R = Arrangement of letters $E, T, T, E, L = \dfrac{\underline{|5}}{\underline{|2}\,\underline{|2}}$

Number of words starting with TE = Arrangement of letters $T, E, R, L = \underline{|4}$

Number of words starting with TL = Arrangement of letters $E, T, E, R = \dfrac{\underline{|4}}{\underline{|2}}$

Number of words starting with TR = Arrangement of letters $T, E, E, L = \dfrac{\underline{|4}}{\underline{|2}}$

$$\text{Rank of TTEERL} = \frac{\underline{|5}}{\underline{|2}} + \frac{\underline{|5}}{\underline{|2}\,\underline{|2}} + \frac{\underline{|5}}{\underline{|2}\,\underline{|2}} + \underline{|4} + \frac{\underline{|4}}{\underline{|2}} + \frac{\underline{|4}}{\underline{|2}} + 2 = 170$$

5.14 TPC-14 : Selection of *r* objects from *n* objects when all *n* objects are not different using 'Integral equation method'

In this problem type, we will have to select r objects from n objects when all n objects are not different. We discussed the same problem type earlier in TPC-9 ...... The method we discussed in TPC-9 .... was the "cases method". *i.e.* we make "cases" based on alike objects in the selection. For example, if we have to select 4 objects, we make cases 'all alike', '3 alike 1 different', '2 alike 2 alike', '2 alike 2 different' and 'all different' cases.

This 'cases' method can be used only if we have to select a few objects say 3, 4, 5. For large number of selection of objects we use 'Integral Equation Method'. In this method, we group alike objects together and with each group we define a variable representing number of objects selected from the group. Then we add all variables and equate the sum to the total objects to be selected.

For example, if we have to select 3 objects from $AAAAABBBBCCC$ objects, then we make groups of identical objects, group of all A objects, group of all B objects and group of all C objects. Let x_1, x_2, x_3 be the number of A, B, C objects selected respectively.

As total number of objects to be selected is 3, we can make following integral equation :

$x_1 + x_2 + x_3 = 3$ [where $0 \le x_i \le 3$ $i = 1, 2, 3$]

Number of solutions of the above integral equation is same as number of ways to select 3 objects from the given objects. This is because every solution of the equation is a way to select 3 objects.

Number of solutions of the equation

$$= \text{Coefficient of } x^{\text{RHS}} \text{ in } \left[x^{\min(x_1)} + x^{\min(x_1)+1} + \ldots\ldots x^{\max(x_1)}\right] \times \left[x^{\min(x_2)} + x^{\min(x_2)+1} + \ldots\ldots x^{\max(x_2)}\right] \times \left[x^{\min(x_3)} + x^{\min(x_3)+1} + \ldots\ldots x^{\max(x_3)}\right]$$

Note : RHS represents right hand side of the equation. For each variable x_1, x_2, x_3, a bracket is formed using the values the variable can take. The derivation of the above method is out of syllabus of the JEE preparation.

$\Rightarrow$ Number of solutions

$= \text{coefficient of } x^3 \text{ in } (x^0 + x^1 + x^2 + x^3)^3$

$= \text{coefficient of } x^3 \text{ in } \left[\dfrac{1-x^4}{1-x}\right]^3 = \text{coefficient of } x^3 \text{ in } (1-x^4)^3 (1-x)^{-3}$

$= \text{coefficient of } x^3 \text{ in } ({}^3C_0 - {}^3C_1 x^4 + {}^3C_2 x^8 - {}^3C_3 x^{12}) (1-x)^{-3}$

$= \text{coefficient of } x^3 \text{ in } (1-x)^{-3}$ [$\because$ other terms cannot generate x^3 term]

$= {}^{3+3-1}C_3 = {}^5C_3 = 10$ [Using: Coefficient of x^r in $(1-x)^{-n} = {}^{n+r-1}C_r$]

Illustration - 37 *In a box there are 10 balls; 4 red, 3 black, 2 white and 1 yellow. In how many ways can a child select 4 balls out of these 10 balls? (Assume that the balls of the same colour are identical)*

(A) 20 **(B)** 18 **(C)** 19 **(D)** 17

SOLUTION : (A)

Let x_1, x_2, x_3 and x_4 be the number of red, black, white, yellow balls selected respectively.

Number of ways to select 4 balls = Number of integral solutions of the equation $x_1 + x_2 + x_3 + x_4 = 4$

Conditions on x_1, x_2, x_3 and x_4

The total number of red, black, white and yellow balls in the box are 4, 3, 2 and 1 respectively.

So we can take : Max $(x_1) = 4$, Max $(x_2) = 3$, Max $(x_3) = 2$, Max $(x_1) = 1$

There is no condition on minimum number of red, black, white and yellow balls selected, so take :

Min $(x_i) = 0$ for $i = 1, 2, 3, 4$

Number of ways to select 4 balls

$= \text{coeff of } x^4 \text{ in } (1 + x + x^2 + x^3 + x^4) \times (1 + x + x^2 + x^3) \times (1 + x + x^2) \times (1 + x)$

$= \text{coeff of } x^4 \text{ in } (1 - x^5)(1 - x^4)(1 - x^3)(1 - x^2)(1 - x)^{-4}$

$= \text{coeff of } x^4 \text{ in } (1 - x)^{-4} - \text{coeff. of } x^2 \text{ in } (1 - x)^{-4} - \text{coeff of } x^1 \text{ in } (1 - x)^{-4} - \text{coeff. of } x^0 \text{ in } (1 - x)^{-4}$

$= {}^7C_4 - {}^5C_2 - {}^4C_1 - {}^3C_0$

$= \dfrac{7 \times 6 \times 5}{3!} - 10 - 4 - 1 = 35 - 15 = 20$

Thus, number of ways of selecting 4 balls from the box subjected to the given conditions is 20.

Another Approach : (using TPC-9 *i.e.* 'cases' method)

The 10 balls are *RRRR BBB WWW Y* (where *R*, *B*, *W*, *Y* represent red, black, white and yellow balls respectively).

The work of selection of the balls from the box can be divided into following categories.

Case - I : All alike

Number of ways of selecting all alike balls $= {}^1C_1 = 1$

Case - II : 3 alike and 1 different

Number of ways of selecting 3 alike and 1 different balls $= {}^2C_1 \times {}^3C_1 = 6$

Case - III : 2 alike and 2 alike

Number of ways of selecting 2 alike and 2 alike balls $= {}^3C_2 = 3$

Case - IV : 2 alike and 2 different

Number of ways of selecting 2 alike and 2 different balls $= {}^3C_1 \times {}^3C_2 = 9$

Case - V : All different

Number of ways of selecting all different balls $= {}^4C_4 = 1$

Total number of ways to select 4 balls = 1 + 6 + 3 + 9 + 1 = 20

5.15 TPC-15 : Points of Intersection between geometrical figures

We can use ${}^nC_r$ (number of ways to select r objects from n different objects) to find points of intersection between geometrical figures.

For example :

(a) Number of points of intersection between n non-concurrent and non parallel lines is ${}^nC_2$.

Logic :

When two lines intersect, we get a point of intersection. Two lines from n different lines can be selected in ${}^nC_2$ ways. Therefore, number of points of intersection is ${}^nC_2$.

(b) Number of lines that can be drawn using n points such that no three of them are collinear is ${}^nC_2$.

Logic :

A line can be drawn through two points. Two points can be selected from n different points in ${}^nC_2$ ways. Therefore, number of lines that can be drawn is ${}^nC_2$.

(c) Number of triangles that can formed using n points such that no three of them are collinear is ${}^nC_3$.

Logic :

A triangle is formed using 3 different points. Three points can be selected from n different points in ${}^nC_3$ ways. Therefore, we can form ${}^nC_3$ triangles using n different points.

(d) Number of diagonals that can be drawn in an n-sided polygon is $\dfrac{n(n-3)}{2}$.

Logic :

There are n vertices in an n-sided polygon. When two vertices are joined (excluding the adjacent vertices), we get a diagonal. The number of ways to select 2 vertices from n vertices is ${}^nC_2$. But this also includes n sides (when adjacent vertices are selected). Therefore number of diagonals

$$= {}^nC_2 - n = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}.$$

Illustrating the Concepts **:**

There are 10 points in a plane, no three of which are in the same straight line, excepting 4 points, which are collinear. Find the (i) number of straight lines obtained from the pairs of these points ; (ii) number of triangles that can be formed with the vertices as these points.

(i) Number of straight lines formed joining the 10 points, taking 2 at a time = ${}^{10}C_2 = \dfrac{10!}{2!8!} = 45$

Number of straight lines formed by joining the four points (which are collinear), taking 2 at a time = ${}^4C_2 = \dfrac{4!}{2!2!} = 6$

But, 4 collinear points, when joined pairwise give only one line.

So, required number of straight lines = 45 – 6 + 1 = 40

(ii) Number of triangles formed by joining the points, taking 3 at a time = ${}^{10}C_3 = \dfrac{10!}{3!7!} = 120$

Number of triangles formed by joining the 4 points (which are collinear), taken 3 at a time = ${}^4C_3 = 4$

Also, 4 collinear points cannot form a triangle when taken 3 at a time

So, required number of triangles = 120 – 4 = 116

Illustration - 38 *In a plane there are 37 straight lines, of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no line passes through both points A and B, and no two are parallel. Find the number of points of intersection of the straight lines.*

(A) 533 **(B)** 536 **(C)** 535 **(D)** 530

SOLUTION : (C)

The number of points of intersection of 37 straight lines is ${}^{37}C_2$. But 13 straight lines out of the given 37 straight lines pass through the same point A. Therefore instead of getting ${}^{13}C_2$ points, we get merely one point A. Similarly, 11 straight lines out of the given 37 straight lines intersect at point B. Therefore instead of getting ${}^{11}C_2$ points, we get only one point B. Hence, the number of instersection points of the lines is ${}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2 = 535$

Illustration - 39 *If m parallel lines in plane are intersected by a family of n parallel lines. Find the number of parallelograms formed.*

(A) $m^2 n^2$ **(B)** $\dfrac{m^2n^2}{4}$ **(C)** $\dfrac{m(m-1)n(n-1)}{4}$ **(D)** $m(m-1)\, n(n-1)$

SOLUTION : (C)

A parallelogram is formed by choosing two straight lines from the set of m parallel lines and two straight lines from the set of n parallel lines.

Two straight lines from the set of m parallel lines can be chosen in ${}^mC_2$ ways and two straight lines from the set of n parallel lines can be chosen in ${}^nC_2$ ways. Hence, the number of parallelograms formed

$$= {}^mC_2 \times {}^nC_2 = \frac{m\,(m-1)}{2} \times \frac{n\,(n-1)}{2} = \frac{mn\,(m-1)\,(n-1)}{4}$$

Illustration - 40 *There are n concurrent lines and another line parallel to one of them. The number of different triangles that will be formed by the (n + 1) lines, is*

(A) $\dfrac{(n-1)n}{2}$ **(B)** $\dfrac{(n-1)(n-2)}{2}$ **(C)** $\dfrac{n\,(n+1)}{2}$ **(D)** $\dfrac{(n+1)\,(n+2)}{2}$

SOLUTION : (B)

The number of triangles = number of selections of 2 lines from the $(n-1)$ lines which are cut by the last line

$$= {}^{n-1}C_2 = \frac{(n-1)!}{2!\,(n-3)!} = \frac{(n-1)\,(n-2)}{2}$$

Illustration - 41 *There are p points in a plane, no three of which are in the same straight line with the exception of q, which are all in the same straight line. Find the number of :*

(i) *straight lines which can be formed by joining them.*

(A) ${}^pC_2 - {}^qC_2$ **(B)** ${}^pC_2 - {}^qC_2 + 1$ **(C)** ${}^pC_2 + {}^qC_2$ **(D)** ${}^pC_2 + {}^qC_2 + 1$

(ii) *triangles which can be formed by joining them.*

(A) ${}^pC_3 - {}^qC_3 + 1$ **(B)** ${}^pC_3 - {}^qC_3 - 1$ **(C)** ${}^pC_3 - {}^qC_3$ **(D)** ${}^pC_3 + {}^qC_3$

SOLUTION : (i).(B) (ii).(C)

(i) If no three of the p points were collinear, the number of straight lines = number of groups of two that can be formed from p points = ${}^pC_2$.

Due to the q points being collinear, there is a loss of ${}^qC_2$ lines that could be formed from these points.

But these points are giving exactly one straight line passing through all of them.

Hence the number of straight lines = ${}^pC_2 - {}^qC_2 + 1$

(ii) If no three points were collinear, the number of triangles = ${}^pC_3$

But there is a loss of ${}^qC_3$ triangles that could be formed from the group of q collinear points.

Hence the number of triangles formed = ${}^pC_3 - {}^qC_3$

5.16 TPC-16 : Formation of Subsets

In this problem type, we select elements from a given set to form subsets. We are supposed to form subsets under constraints. For example, two subsets P and Q are to be formed such that $P \cup Q$ has all elements, $P \cap Q$ has no elements *etc.* To understand the problems based on this type, read the following illustrations carefully.

Illustration - 42 *Let X is a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random then find the number of ways to form sets such that $P \cup Q = X$.*

(A) 3^n **(B)** 2^n **(C)** $2^n - 1$ **(D)** $3^n - 1$

SOLUTION : (A)

As $P \cup Q = X$, it means every element would be either included in P or in Q or both. So for every element, there are 3 choices.

$\Rightarrow$ Number of ways to select P and Q such that $(P \cup Q = X) = 3^n$

Illustration - 43 *Let X is a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random. Find number of ways to choose P and Q such that $P \cup Q$ contains exactly r elements.*

(A) 3^r **(B)** ${}^nC_r\, 3^r$ **(C)** 3^n **(D)** 2^n

SOLUTION : (B)

$P \cup Q$ has r elements. It means r elements out of n elements should be present in either P or in Q or both. r elements out of n elements can be selected in ways = ${}^nC_r$

Each of these r elements has 3 choices

$\Rightarrow$ No. of ways to select elements for P and $Q = 3^r$

Rest $(n - r)$ elements has 1 choice *i.e.* neither go in P nor in Q $\Rightarrow$ No. of ways = 1^{n-r}.

$\Rightarrow$ Number of ways to select P and Q such that $P \cup Q$ has exactly r elements = ${}^nC_r\, 3^r\, (1)^{n-r} = {}^nC_r\, 3^r$

Illustration - 44 *Let X is a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random. Find number of ways to select P and Q such that $P \cap Q$ is empty i.e. $P \cap Q = \phi$.*

(A) 3^n (B) 2^n (C) $2^n - 1$ (D) $3^n - 1$

SOLUTION : (A)

$P \cap Q = \phi$. It means P and Q should be disjoint sets. That is there is no element common in P and Q.

$\Rightarrow$ For every elements in set X there are 3 choices. Either it is selected in P but not in Q or selected in Q but not in P or not selected in both P and Q.

$\Rightarrow$ Number of ways to select P and Q such that $P \cap Q$ is $\phi = 3^n$

Illustration - 45 *Let X is a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random. Find number of ways to select P and Q such that $P = \bar{Q}$.*

(A) 3^n (B) 2^n (C) $2^n - 1$ (D) $3^n - 1$

SOLUTION : (B)

$P = \bar{Q}$. It means P and Q are complementary sets i.e. every element present in X is either present in P or Q

$\Rightarrow$ For every element there are 2 choices to select. Either it will be selected for P or it will be selected for Q.

$\Rightarrow$ No. of ways to select $= 2^n$

DIVISION AND DISTRIBUTION OF NON-IDENTICAL ITEMS Section - 6

6.1 Case - I : Unequal division and distribution of non-identical objects

In this section, we will discuss ways to divide non-identical objects into groups. For example, if we have to divide three different balls (b_1, b_2, b_3) among 2 boys (B_1 and B_2) such that B_1 gets 2 balls and B_2 gets 1 ball, then

Number of ways to divide balls among boys is 3 ways as shown in the following table.

B_1	B_2
b_1, b_2	b_3
b_2, b_3	b_1
b_3, b_1	b_2

Instead of writing all ways and counting them, we can make a formula to find number of ways.

First select 2 balls for B_1 in ${}^3C_2$ and then remaining 1 ball for B_2 in ${}^1C_1$ ways.

Total number of ways, using fundamental principle of counting, is

$$= {}^3C_2 \times {}^1C_1 = 3 \times 1 = 3 \text{ ways.}$$